

Asymptotics of Sample Entropy Production Rate for Stochastic Differential Equations*

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October 8, 2015

Abstract

By using the dimension-free Harnack inequality and the integration by parts formula for the associated diffusion semigroup, we prove the central limit theorem, the moderate deviation principle, and the logarithmic iteration law for the sample entropy production rate of stochastic differential equations with Lipschitz continuous and dissipative drifts.

AMS subject Classification: 65G17, 65G60.

Keywords: Sample entropy production rate, central limit theorem, moderate deviation principle, logarithmic iteration law, stochastic differential equation.

1 Introduction

The entropy production rate (EPR in short) is a key element of the second law of thermodynamics for open systems, see for instance [8, 10, 6, 13, 14]. In this paper we characterize the asymptotic behaviors of the sample EPR for diffusion processes.

*F.Y. Wang is supported in part by NNSFC(11131003, 11431014), the 985 project and the Laboratory of Mathematical and Complex Systems. J. Xiong was supported by Macao Science and Technology Fund FDCT 076/2012/A3 and Multi-Year Research Grants of the University of Macau Nos. MYRG2014-00015-FST and MYRG2014-00034-FST. L. Xu is supported by the grant Science and Technology Development Fund, Macao S.A.R FDCT 049/2014/A1 and the grant MYRG2015-00021-FST.

Let $(X_t)_{t \geq 0}$ be a stationary diffusion process on \mathbb{R}^d with invariant probability measure μ . It is called reversible if $X_{[0,t]} := (X_r)_{0 \leq r \leq t}$ and the reverse $\bar{X}_{[0,t]} := (X_{t-r})_{0 \leq r \leq t}$ are identified in distributions for all $t > 0$. In case that the process is not reversible, the sample EPR is an important object to measure the difference between the distributions of the process and its reverse. More precisely, for $\mathbb{P}_{[0,t]}$ and $\bar{\mathbb{P}}_{[0,t]}$ being the distributions of $X_{[0,t]}$ and $\bar{X}_{[0,t]}$ respectively, the sample EPR of the process is defined as (see [11])

$$\mathcal{R}_t(X_{[0,t]}) = \frac{1}{t} \log \frac{d\mathbb{P}_{[0,t]}}{d\bar{\mathbb{P}}_{[0,t]}}, \quad t > 0,$$

which is a measurable function on $C([0,t]; \mathbb{R}^d)$ for every $t > 0$. If $\mathbb{P}_{[0,t]}$ is not absolutely continuous with respect to $\bar{\mathbb{P}}_{[0,t]}$, we set $\mathcal{R}_t(X_{[0,t]}) = \infty$. It is well known that $\mathcal{R} := \mathbb{E}\mathcal{R}_t(X_{[0,t]})$ is non-negative and independent of t , and when it is finite we have

$$(1.1) \quad \lim_{t \rightarrow \infty} \mathcal{R}_t(X_{[0,t]}) = \mathcal{R} \quad \text{a.s.}$$

according to the ergodic theorem. The purpose of this paper is to investigate long time behaviors of $\mathcal{R}_t(X_{[0,t]})$, which include the central limit theorem (CLT in short), the moderate deviation principle (MDP in short) and the logarithmic iteration law (LIL in short).

Consider the following stochastic differential equation (SDE in short) on \mathbb{R}^d :

$$(1.2) \quad dX_t = B(X_t)dt + \sigma dW_t,$$

where W_t is the d -dimensional Brownian motion on a complete filtration probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, σ is an invertible $d \times d$ -matrix, and $B : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Lipschitz continuous so that ∇B exists with $\|\nabla B\|_\infty < \infty$. We further assume that B satisfies the dissipativity condition

$$(1.3) \quad \langle B(x) - B(y), x - y \rangle \leq \kappa|x - y| - K|x - y|^2, \quad x, y \in \mathbb{R}^d$$

for some constants $\kappa \geq 0, K > 0$. Note that (1.3) holds for $B := B_0 + B_1$ where B_0 is bounded and $B_1 \in C^1$ such that $\langle \nabla_v B_1(x), v \rangle \leq -K|v|^2$ for $x, v \in \mathbb{R}^d$. It is well known that in this situation the SDE (1.2) has a unique non-explosive solution for any initial distributions, and the associate Markov semigroup P_t has a unique invariant probability measure μ . According to [3], we have $\mu(dx) = \rho(x)dx$ for some strictly positive density function $\rho \in \cap_{p>1} W_{loc}^{p,1}(dx)$, see Proposition 2.1 below for details. Throughout the paper, we denote $\nu(f) = \int_{\mathbb{R}^d} f d\nu$ for a measure ν and $f \in L^1(\nu)$.

We now formulate the sample EPR for the solution to (1.2). It is well known that the reverse process is a weak solution to the SDE (see e.g. [11, Theorem 3.3.5])

$$(1.4) \quad d\bar{X}_t = \{\sigma\sigma^* \nabla \log \rho(\bar{X}_t) - B(\bar{X}_t)\}dt + \sigma dW_t.$$

Since the drift is in $L_{loc}^p(dx)$ for all $p > 1$, according to [21] this SDE has a unique solution for any initial point. We will prove

$$(1.5) \quad \mu(\exp[\varepsilon(|B|^2 + |\nabla \log \rho|^2)]) < \infty \quad \text{for some constant } \varepsilon > 0$$

and that the process

$$(1.6) \quad M_t := \exp \left[\int_0^t \langle \sigma^* \nabla \log \rho - 2\sigma^{-1}B, dW_s \rangle - \frac{1}{2} \int_0^t |\sigma^* \nabla \log \rho - 2\sigma^{-1}B|^2 ds \right], \quad t \geq 0$$

is a martingale (see Proposition 2.1 below). Then by the Girsanov theorem,

$$\bar{W}_s := W_s + \int_0^s \{2\sigma^{-1}B(X_u) - \sigma^* \nabla \log \rho(X_u)\} du, \quad s \in [0, t]$$

is a d -dimensional Brownian motion under the probability $d\mathbb{Q}_t := M_t d\mathbb{P}$. Reformulating (1.2) as

$$dX_t = \{\sigma \sigma^* \nabla \log \rho(X_t) - B(X_t)\} dt + \sigma d\bar{W}_t,$$

we see that the solution to (1.4) is non-explosive and, by the weak uniqueness, we obtain

$$\mathbb{E}f(\bar{X}_{[0,t]}) = \mathbb{E}_{\mathbb{Q}_t}f(X_{[0,t]}) = \mathbb{E}\{M_t f(X_{[0,t]})\}, \quad F \in \mathcal{B}_b(C([0, t]; \mathbb{R}^d)).$$

This implies $\frac{d\bar{\mathbb{P}}_{[0,t]}}{d\mathbb{P}_{[0,t]}}(X_{[0,t]}) = M_t$, so that the sample EPR of X_t can be formulated as

$$(1.7) \quad \begin{aligned} \mathcal{R}_t(X_{[0,t]}) &= \frac{1}{t} \log \frac{1}{M_t} \\ &= -\frac{1}{t} \int_0^t \langle \sigma^* \nabla \log \rho - 2\sigma^{-1}B, dW_s \rangle + \frac{1}{2t} \int_0^t |\sigma^* \nabla \log \rho - 2\sigma^{-1}B|^2 ds. \end{aligned}$$

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of probability measures on \mathbb{R}^d . For any $\nu \in \mathcal{P}(\mathbb{R}^d)$, let $(X_t^\nu)_{t \geq 0}$ be the solution to (1.2) with initial distribution ν . When $\nu = \delta_x$, the Dirac measure at point x , we simply denote X^ν by X^x . Let

$$U_\mu^p(l) = \left\{ \nu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \left(\frac{d\nu}{d\mu} \right)^p d\mu \leq l \right\}, \quad p > 1, l > 0.$$

The main result of the paper is the following, which includes CLT, MDP and LIL for the sample EPR process $\mathcal{R}_t(X_{[0,t]})$.

Theorem 1.1. *Assume that B is Lipschitz continuous and (1.3) holds for some constants $\kappa \geq 0$ and $K > 0$. Then the following assertions hold:*

- (1) (1.5) holds, and $\delta := \lim_{t \rightarrow \infty} t \mathbb{E}\{\mathcal{R}_t(X_{[0,t]}^\mu) - \mathcal{R}\}^2 < \infty$ exists.
- (2) **(CLT)** For any $p > 1$ and $l > 0$, $\lim_{t \rightarrow \infty} \mathbb{P}(\sqrt{t}\{\mathcal{R}_t(X_{[0,t]}^\nu) - \mathcal{R}\} \in \cdot) = N(0, \delta)$ weakly and uniformly in $\nu \in U_\mu^p(l)$, where $N(0, \delta)$ is the centered Gaussian distribution with variance δ .

- (3) **(MDP)** For any $\lambda : (0, \infty) \rightarrow (0, \infty)$ with $\lambda(t) \wedge \frac{\sqrt{t}}{\lambda(t)} \rightarrow \infty$ as $t \rightarrow \infty$, any measurable set $A \subset \mathbb{R}$ and constants $p > 1, l > 0$,

$$\begin{aligned} -\inf_{u \in A^\circ} \frac{u^2}{2\delta} &\leq \liminf_{t \rightarrow \infty} \frac{1}{\lambda(t)^2} \log \mathbb{P} \left(\frac{\sqrt{t}}{\lambda(t)} \{ \mathcal{R}_t(X_{[0,t]}^\nu) - \mathcal{R} \} \in A \right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{\lambda(t)^2} \log \mathbb{P} \left(\frac{\sqrt{t}}{\lambda(t)} \{ \mathcal{R}_t(X_{[0,t]}^\nu) - \mathcal{R} \} \in A \right) \leq -\inf_{u \in \bar{A}} \frac{u^2}{2\delta} \end{aligned}$$

holds uniformly in $\nu \in U_\mu^p(l)$, where A° and \bar{A} are the interior and closure of A respectively.

- (4) **(LIL)** For any $\nu \in \mathcal{P}(\mathbb{R}^d)$ with $\frac{d\nu}{d\mu} \in L^p(\mu)$ for some $p > 1$, \mathbb{P} -a.s.

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\sqrt{t}}{\sqrt{2 \log \log t}} \{ \mathcal{R}_t(X_{[0,t]}^\nu) - \mathcal{R} \} &= \sqrt{\delta}, \\ \liminf_{t \rightarrow \infty} \frac{\sqrt{t}}{\sqrt{2 \log \log t}} \{ \mathcal{R}_t(X_{[0,t]}^\nu) - \mathcal{R} \} &= -\sqrt{\delta}. \end{aligned}$$

Since the rate function in Theorem 1.1(3) is continuous, for any domain $A \subset \mathbb{R}$ we have

$$\lim_{t \rightarrow \infty} \frac{1}{\lambda(t)^2} \log \mathbb{P} \left(\frac{\sqrt{t}}{\lambda(t)} \{ \mathcal{R}_t(X_{[0,t]}^\nu) - \mathcal{R} \} \in A \right) = -\inf_{u \in A} \frac{u^2}{2\delta}.$$

The next result extends Theorem 1.1(2)-(4) to $\nu = \delta_x$, the Dirac measure at point x , which is singular with respect to μ .

Theorem 1.2. *In the situation of Theorem 1.1, the following assertions hold.*

- (1) $\lim_{t \rightarrow \infty} \mathbb{P}(\sqrt{t} \{ \mathcal{R}_t(X_{[0,t]}^x) - \mathcal{R} \} \in \cdot) = N(0, \delta)$ weakly and locally uniformly in $x \in \mathbb{R}^d$.
(2) For any $\lambda : (0, \infty) \rightarrow (0, \infty)$ with $\lambda(t) \wedge \frac{\sqrt{t}}{\lambda(t)} \rightarrow \infty$ as $t \rightarrow \infty$, and any measurable set $A \subset \mathbb{R}$,

$$\begin{aligned} -\inf_{u \in A^\circ} \frac{u^2}{2\delta} &\leq \liminf_{t \rightarrow \infty} \frac{1}{\lambda(t)^2} \log \mathbb{P} \left(\frac{\sqrt{t}}{\lambda(t)} \{ \mathcal{R}_t(X_{[0,t]}^x) - \mathcal{R} \} \in A \right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{\lambda(t)^2} \log \mathbb{P} \left(\frac{\sqrt{t}}{\lambda(t)} \{ \mathcal{R}_t(X_{[0,t]}^x) - \mathcal{R} \} \in A \right) \leq -\inf_{u \in \bar{A}} \frac{u^2}{2\delta} \end{aligned}$$

holds locally uniformly in $x \in \mathbb{R}^d$.

- (3) For any $x \in \mathbb{R}^d$, \mathbb{P} -a.s.

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\sqrt{t}}{\sqrt{2 \log \log t}} \{ \mathcal{R}_t(X_{[0,t]}^x) - \mathcal{R} \} &= \sqrt{\delta}, \\ \liminf_{t \rightarrow \infty} \frac{\sqrt{t}}{\sqrt{2 \log \log t}} \{ \mathcal{R}_t(X_{[0,t]}^x) - \mathcal{R} \} &= -\sqrt{\delta}. \end{aligned}$$

We will prove the above two results in Section 3, for which some preparations are presented in Sections 2. Finally, SDEs with multiplicative noise are discussed in Section 4.

2 Preparations

Let P_t be the Markov semigroup associated to the SDE (1.2), and let

$$\mathcal{U} := \{\rho_\nu d\mu : \rho_\nu \geq 0, \mu(\rho_\nu) = 1, \mu(\rho_\nu^p) < \infty \text{ for some } p > 1\}.$$

The main result of this section is the following.

Proposition 2.1. *Assume that B is Lipschitz continuous and (1.3) holds for some constants $\kappa \geq 0$ and $K > 0$. Then:*

- (1) P_t has a unique invariant probability measure μ , which has strictly positive density $\rho \in \cap_{p>1} W_{loc}^{p,1}(dx)$, and $\mu(e^{\varepsilon(|\cdot|^2 + |\nabla \log \rho|^2)}) < \infty$ holds for some constant $\varepsilon > 0$.
- (2) The density $p_t(x, y)$ of P_t with respect to μ satisfies

$$(2.1) \quad \mu(p_t(x, \cdot)^q) \leq \exp \left[2q(q-1)\kappa^2 \|\sigma^{-1}\|^2 + \frac{4q(q-1)K \|\sigma^{-1}\|^2 (\mu(|\cdot|^2) + |x|^2)}{e^{2Kt} - 1} \right]$$

for all $q > 1, t > 0, x \in \mathbb{R}^d$. Consequently, P_t is hyperbounded, i.e. $\|P_t\|_{L^2(\mu) \rightarrow L^4(\mu)} < \infty$ for some $t > 0$.

- (3) For any $t > 0$, 1 is a simple eigenvalue of P_t .
- (4) For any $p > 1 \vee \frac{d}{2}$ there exist a constant $c > 0$ and a positive function $H \in C(\mathbb{R}^d)$ such that

$$\int_0^t P_s |f|(x) ds \leq \mu(|f|^p)^{\frac{1}{p}} H(x) \left(t + t^{\frac{2p-d}{2p}} \right), \quad x \in \mathbb{R}^d, t \geq 0, f \in L^p(\mu).$$

- (5) If $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable such that $\mu(e^{\varepsilon|\psi|^2}) < \infty$ for some $\varepsilon > 0$, then for any $\nu \in \{\delta_x : x \in \mathbb{R}^d\} \cup \mathcal{U}$,

$$M_t^\nu := \exp \left[\int_0^t \langle \psi(X_s^\nu), dW_s \rangle - \frac{1}{2} \int_0^t |\psi(X_s^\nu)|^2 ds \right], \quad t \geq 0$$

is a martingale.

To prove this result we need the following lemma on exponential integrability, integration by parts formula and Harnack inequality.

Lemma 2.2. *Assume that B is Lipschitz continuous and (1.3) holds for some constants $\kappa \geq 0$ and $K > 0$. Then:*

- (1) There exist constants $\varepsilon, c > 0$ such that

$$(2.2) \quad \mathbb{E} \int_0^t e^{\varepsilon |X_s^x|^2} ds \leq c(t + e^{\varepsilon |x|^2}), \quad x \in \mathbb{R}^d, t \geq 0.$$

(2) For any $f \in \mathcal{B}_b^+(\mathbb{R}^d)$, $T > 0, p > 1$ and $x, y \in \mathbb{R}^d$,

$$(2.3) \quad (P_T f(x))^p \leq (P_T f^p(y)) \exp \left[\frac{2p\kappa^2 \|\sigma^{-1}\|^2 (e^{KT} - 1)}{(p-1)(e^{KT} + 1)} + \frac{2pK \|\sigma^{-1}\|^2 |x - y|^2}{(p-1)(e^{2KT} - 1)} \right].$$

(3) For any $f \in C_b^1(\mathbb{R}^d)$,

$$(2.4) \quad P_T \nabla f(x) = \mathbb{E} \left[\frac{f(X_T^x)}{T} \int_0^T \{ \sigma^{-1} (I - t \nabla B(X_t^x)) \}^* dW_t \right], \quad T > 0, x \in \mathbb{R}^d.$$

Proof. Assertion (1) follows from condition (1.3) by Itô's formula, and the other two can be easily proved by using coupling by change of measures as in [16]. We include below brief proofs of these assertions for completeness.

(1) It is easy to see that the generator of the diffusion process is

$$L = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij} \partial_i \partial_j + \sum_{i=1}^d B_i(x) \partial_i.$$

Then condition (1.3) implies that for small enough $\varepsilon > 0$,

$$(2.5) \quad L e^{\varepsilon |\cdot|^2}(x) \leq C_1 - C_2 e^{\varepsilon |x|^2}, \quad x \in \mathbb{R}^d$$

holds for some constants $C_1, C_2 > 0$. So, by Itô's formula we obtain

$$C_2 \mathbb{E} \int_0^t e^{\varepsilon |X_s^x|^2} ds \leq e^{\varepsilon |x|^2} + C_1 t,$$

which implies (2.2) for $c := \frac{1 \vee C_1}{C_2}$.

(2) For fixed $x, y \in \mathbb{R}^d$ and $T > 0$, let $X_t = X_t^x$ solve (1.2) for $X_0 = x$, and construct Y_t with $Y_0 = y$ as follows. For

$$\xi_t := \kappa e^{-K(T-t)} + \frac{2K e^{Kt} |x - y|}{e^{2KT} - 1}, \quad t \geq 0,$$

the SDE

$$dY_t = \left\{ B(Y_t) + \xi_t \frac{X_t - Y_t}{|X_t - Y_t|} \right\} dt + \sigma dW_t, \quad Y_0 = y$$

has a unique solution before the coupling time

$$\tau := \inf \{ t \geq 0 : X_t = Y_t \}.$$

Take $Y_t = X_t$ for $t \geq \tau$. Then $(Y_t)_{t \geq 0}$ solves the SDE

$$(2.6) \quad dY_t = B(Y_t) dt + \sigma d\tilde{W}_t, \quad Y_0 = y$$

for

$$\tilde{W}_t := W_t + \int_0^{t \wedge \tau} \xi_s \frac{\sigma^{-1}(X_s - Y_s)}{|X_s - Y_s|} ds, \quad s \geq 0.$$

By the Girsanov theorem, $(\tilde{W}_t)_{t \in [0, T]}$ is a d -dimensional Brownian motion under the probability $\mathbb{Q} := R\mathbb{P}$ for

$$R := \exp \left[- \int_0^{T \wedge \tau} \left\langle \xi_s \frac{\sigma^{-1}(X_s - Y_s)}{|X_s - Y_s|}, dW_s \right\rangle - \frac{1}{2} \int_0^{T \wedge \tau} \left| \xi_s \frac{\sigma^{-1}(X_s - Y_s)}{|X_s - Y_s|} \right|^2 ds \right].$$

Combining (2.6) with (1.2) and using condition (1.3), we obtain

$$d|X_t - Y_t| \leq (\kappa - \xi_t - K|X_t - Y_t|)dt, \quad t \leq \tau \wedge T.$$

This together with the definition of ξ_t leads to

$$\begin{aligned} |X_t - Y_t| &\leq e^{-Kt}|x - y| + \int_0^t e^{-K(t-s)}(\kappa - \xi_s)ds \\ &= \frac{\kappa e^{-Kt}}{K} \left(e^{Kt} - 1 - \frac{e^{2Kt} - 1}{e^{KT} + 1} \right) + |x - y|e^{-Kt} \left(1 - \frac{e^{2Kt} - 1}{e^{2KT} - 1} \right), \quad t \in [0, T \wedge \tau]. \end{aligned}$$

So, if $\tau > T$ then by the definition of τ we have

$$0 < |X_T - Y_T| \leq \frac{\kappa e^{-KT}}{K} \left(e^{KT} - 1 - \frac{e^{2KT} - 1}{e^{KT} + 1} \right) + |x - y|e^{-KT} \left(1 - \frac{e^{2KT} - 1}{e^{2KT} - 1} \right) = 0,$$

which is impossible. Therefore, $\tau \leq T$ a.s. so that $X_T = Y_T$. Combining this with (2.6) and noting that \tilde{W}_t is a Brownian motion under $R\mathbb{P}$, for $f \in \mathcal{B}_b^+(\mathbb{R}^d)$ we have

$$(P_T f(y))^p = (\mathbb{E}\{Rf(Y_T)\})^p = (\mathbb{E}\{Rf(X_T)\})^p \leq \{\mathbb{E}f^p(X_T)\}(\mathbb{E}R^{\frac{p}{p-1}})^{p-1}$$

and

$$\begin{aligned} \mathbb{E}R^{\frac{p}{p-1}} &\leq \mathbb{E} \exp \left[\frac{p\|\sigma^{-1}\|^2}{2(p-1)^2} \int_0^T |\xi_s|^2 ds - \frac{p^2}{2(p-1)^2} \int_0^{T \wedge \tau} \left| \xi_s \frac{\sigma^{-1}(X_s - Y_s)}{|X_s - Y_s|} \right|^2 ds \right. \\ &\quad \left. - \frac{p}{p-1} \int_0^{T \wedge \tau} \left\langle \xi_s \frac{\sigma^{-1}(X_s - Y_s)}{|X_s - Y_s|}, dW_s \right\rangle \right] \\ &= \exp \left[\frac{p\|\sigma^{-1}\|^2}{2(p-1)^2} \int_0^T |\xi_s|^2 ds \right] \\ &\leq \exp \left[\frac{2p\kappa^2\|\sigma^{-1}\|^2(e^{KT} - 1)}{(p-1)^2(e^{KT} + 1)} + \frac{2pK\|\sigma^{-1}\|^2|x - y|^2}{(p-1)^2(e^{2KT} - 1)} \right]. \end{aligned}$$

Hence,

$$(P_T f(y))^p \leq (P_T f^p(x)) \exp \left[\frac{2p\kappa^2\|\sigma^{-1}\|^2(e^{KT} - 1)}{(p-1)(e^{KT} + 1)} + \frac{2pK\|\sigma^{-1}\|^2|x - y|^2}{(p-1)(e^{2KT} - 1)} \right].$$

(3) Again let $X_t = X_t^x$. For any $v \in \mathbb{R}^d$ with $|v| = 1$ and $r \in [0, 1]$, let Y_t^r solve the SDE

$$dY_t^r = \left\{ B(X_t) + \frac{rv}{T} \right\} dt + \sigma dW_t, \quad Y_0^r = x.$$

Then

$$(2.7) \quad Y_t^r - X_t = \frac{rtv}{T}, \quad t \in [0, T].$$

Since B is Lipschitz continuous, we have

$$|B(Y_t^r) - B(X_t)| \leq \|\nabla B\|_\infty |Y_t^r - X_t| \leq \|\nabla B\|_\infty r|v| < \infty, \quad t \in [0, T], r \in [0, 1].$$

Then by the Girsanov theorem,

$$W_t^r := W_t + \int_0^t \sigma^{-1} \left\{ B(X_s) - B(Y_s^r) + \frac{rv}{T} \right\} ds, \quad s \in [0, T]$$

is a d -dimensional Brownian motion under the probability $R_r \mathbb{P}$, where

$$R_r := \exp \left[- \int_0^T \left\langle \sigma^{-1} \left\{ B(X_s) - B(Y_s^r) + \frac{rv}{T} \right\}, dW_s \right\rangle - \frac{1}{2} \int_0^T \left| \sigma^{-1} \left\{ B(X_s) - B(Y_s^r) + \frac{rv}{T} \right\} \right|^2 ds \right].$$

Combining this with $Y_T^r = X_T + rv$ due to (2.7), we obtain

$$P_t f(x) = \mathbb{E}\{R_r f(Y_T^r)\} = \mathbb{E}\{R_r f(X_T + rv)\}, \quad r \in [0, 1].$$

Due to (2.7), $\|\nabla B\|_\infty < \infty$ and the definition of R_r , for any $f \in C_b^1(\mathbb{R}^d)$ we may take derivative for both sides in r at $r = 0$ to derive

$$\begin{aligned} 0 &= \frac{d}{dr} \Big|_{r=0} P_T f(x) = \mathbb{E} \left\{ f(X_T) \frac{dR_r}{dr} \Big|_{r=0} \right\} + \mathbb{E} \{ (\nabla_v f)(X_T) \} \\ &= \frac{1}{T} \mathbb{E} \left\{ f(X_T) \int_0^T \langle \sigma^{-1} \{ t \nabla_v B(X_s) - v \}, dW_s \rangle \right\} + P_T (\nabla_v f)(x), \quad v \in \mathbb{R}^d. \end{aligned}$$

This implies (2.4) □

Proof of Proposition 2.1. (1) It is well known that (2.5) implies the existence of invariant probability measure and that any invariant probability measure μ satisfies $\mu(e^{\varepsilon|\cdot|^2}) < \infty$. By the Harnack inequality (2.3), μ is the unique invariant probability measure (see [16, Theorem 1.4.1(3)] or [19, Proposition 3.1]). As already indicated in the Introduction that according to [3], $\mu(dx) = \rho(x)dx$ holds for some strictly positive $\rho \in \cap_{p>1} W_{loc}^{p,1}(dx)$. It remains to prove that $\mu(e^{\varepsilon|\nabla \log \rho|^2}) < \infty$ for some $\varepsilon > 0$.

Let X_t^μ be the solution to (1.2) with initial distribution μ . Since μ is P_t -invariant, by taking integral for (2.4) with respect to $\mu(dx)$ we obtain

$$\begin{aligned} \mu(\nabla f) &= \mu(P_1 \nabla f) = \mathbb{E} \left\{ f(X_1^\mu) \int_0^1 \{ \sigma^{-1} (I - t \nabla B(X_t^\mu)) \}^* dW_t \right\} \\ &= \mathbb{E} \left\{ f(X_1^\mu) \mathbb{E} \left(\int_0^1 \{ I - \sigma^{-1} (t \nabla B(X_t^\mu)) \}^* dW_t \Big| X_1^\mu \right) \right\}, \quad f \in C_0^1(\mathbb{R}^d). \end{aligned}$$

On the other hand, by the integration by parts formula for the Lebesgue measure, for any $f \in C_0^1(\mathbb{R}^d)$ we have

$$\mu(\nabla f) = -\mu(f \nabla \log \rho) = -\mathbb{E}\{f(X_1^\mu) \nabla \log \rho(X_1^\mu)\}.$$

Combining this with the above display we obtain

$$\nabla \log \rho(X_1^\mu) = \mathbb{E}\left(\int_0^1 [\sigma^{-1}(t \nabla B(X_t^\mu) - I)]^* dW_t \middle| X_1^\mu\right), \quad \text{a.s.}$$

Then by Jensen's inequality and noting that $\|\nabla B\|_\infty < \infty$, we have

$$\begin{aligned} \mu(e^{\varepsilon |\nabla \log \rho|^2}) &= \mathbb{E} \exp [\varepsilon |\nabla \log \rho(X_1^\mu)|^2] \\ &\leq \mathbb{E} \left\{ \mathbb{E} \left[\exp \left(\varepsilon \left| \int_0^1 [\sigma^{-1}(t \nabla B(X_t^\mu) - I)]^* dW_t \right|^2 \right) \middle| X_1^\mu \right] \right\} \\ &= \mathbb{E} \exp \left[\varepsilon \left| \int_0^1 \{\sigma^{-1}(t \nabla B(X_t^\mu) - I)\}^* dW_t \right|^2 \right] < \infty \end{aligned}$$

for small enough $\varepsilon > 0$.

(2) By the Harnack inequality (2.3), for $x, y \in \mathbb{R}^d$ and $t > 0$ we have

$$\left| \int_{\mathbb{R}^d} p_t(x, z) f(z) \mu(dz) \right|^{\frac{q}{q-1}} \exp \left[-\frac{2qK \|\sigma^{-1}\|^2 |x - y|^2}{e^{2Kt} - 1} \right] \leq \exp [2q\kappa^2 \|\sigma^{-1}\|^2] P_t |f|^{\frac{q}{q-1}}(y).$$

Taking integral with respect to $\mu(dy)$, when $\mu(|f|^{\frac{q}{q-1}}) = 1$ we obtain

$$\left| \int_{\mathbb{R}^d} p_t(x, z) f(z) \mu(dz) \right|^{\frac{q}{q-1}} \leq \frac{\exp[2q\kappa^2 \|\sigma^{-1}\|^2]}{\mu \left(\exp \left[-\frac{2qK \|\sigma^{-1}\|^2 |x - \cdot|^2}{e^{2Kt} - 1} \right] \right)}.$$

This implies

$$(2.8) \quad \mu(p_t(x, \cdot)^q) \leq \left\{ \frac{\exp[2q\kappa^2 \|\sigma^{-1}\|^2]}{\mu \left(\exp \left[-\frac{2qK \|\sigma^{-1}\|^2 |x - \cdot|^2}{e^{2Kt} - 1} \right] \right)} \right\}^{q-1}.$$

By Jensen's inequality we have

$$\begin{aligned} \mu \left(\exp \left[-\frac{2qK \|\sigma^{-1}\|^2 |x - \cdot|^2}{e^{2Kt} - 1} \right] \right) &\geq \exp \left[-\mu \left(\frac{2qK \|\sigma^{-1}\|^2 |x - \cdot|^2}{e^{2Kt} - 1} \right) \right] \\ &\geq \exp \left[-\frac{4qK \|\sigma^{-1}\|^2 (|x|^2 + \mu(|\cdot|^2))}{e^{2Kt} - 1} \right]. \end{aligned}$$

Substituting this into (2.8) we prove (2.1).

Next, by (2.1), there exists a constant $C > 0$ such that if $\mu(f^2) \leq 1$ then

$$|P_t f(x)|^4 \leq [\mu(p_t(x, \cdot)^2)]^2 [\mu(f^2)]^2 \leq \mu(p_t(x, \cdot)^4) \leq e^{\frac{C(1+t+|x|^2)}{t}}.$$

Since $\mu(e^{\varepsilon|\cdot|^2}) < \infty$ for some constant $\varepsilon > 0$, when $t > 0$ is large enough this implies

$$\sup_{\mu(f^2) \leq 1} \mu((P_t f)^4) \leq e^{\frac{C(1+t)}{t}} \mu(e^{\frac{C|\cdot|^2}{t}}) < \infty.$$

Thus, P_t is hyperbounded.

(3) Since due to (1) we have $|\nabla \log \rho| + |B| \in L^p(\mu)$ for any $p > 1$, by [12, Proposition 2.11] with $c = b = 0$ and $d = B - \sigma \sigma^* \nabla \log \rho$ which is divergence free as μ is an invariant probability measure, P_t is associated to a Dirichlet form with symmetric part

$$\mathcal{E}(f, g) := \mu(\langle \sigma \sigma^* \nabla f, \nabla g \rangle), \quad f, g \in H_\sigma^{2,1}(\mu),$$

where $H_\sigma^{2,1}(\mu)$ is the completion of $C_0^\infty(\mathbb{R}^d)$ under the Sobolev norm

$$\|f\|_{2,1} := \sqrt{\mu(f^2) + \mathcal{E}(f, f)}.$$

Obviously, the Dirichlet form is irreducible so that $P_t \rightarrow \mu$ in $L^2(\mu)$ as $t \rightarrow \infty$. If $f \in L^2(\mu)$ such that $P_t f = f$ for some $t > 0$, then $f = P_{nt} f \rightarrow \mu(f)$ in $L^2(\mu)$ as $n \rightarrow \infty$, so that f has to be constant. Thus, 1 is a simple eigenvalue of P_t .

(4) By the Harnack inequality (2.3) we obtain

$$(2.9) \quad (P_s |f|(x))^p \int_{\mathbb{R}^d} e^{-c-c|x-y|^2/s} \mu(dy) \leq \mu(|f|^p)$$

for some constant $c = c(p) > 0$. Since $\rho \in C(\mathbb{R}^d)$ is strictly positive as already explained in Introduction due to [3], we have $\mu(B(x, \sqrt{s})) \geq (1 \wedge s)^{d/2} h_1(x)$ for some strictly positive $h_1 \in C(\mathbb{R}^d)$ and all $s \geq 0, x \in \mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} e^{-c-c|x-y|^2/s} \mu(dy) \geq e^{-c} \mu(B(x, \sqrt{s})) \geq e^{-c} (1 \wedge s)^{d/2} h_1(x), \quad s > 0, x \in \mathbb{R}^d.$$

Combining this with (2.9) we obtain

$$P_s |f|(x) \leq \mu(|f|^p)^{\frac{1}{p}} (1 \wedge s)^{-\frac{d}{2p}} h_2(x), \quad s > 0, x \in \mathbb{R}^d$$

for some positive $h_2 \in C(\mathbb{R}^d)$. Since $p > \frac{d}{2}$, this implies the desired estimate.

(5) By (4) and $\mu(e^{\varepsilon|\psi|^2}) < \infty$ we have $\mathbb{E} \int_0^t |\psi(X_s^x)|^2 ds < \infty$ for any $t > 0$ and $x \in \mathbb{R}^d$, and for $\rho_\nu := \frac{d\nu}{d\mu} \in L^q(\mu)$,

$$\begin{aligned} \mathbb{E} \int_0^t |\psi(X_s^\nu)|^2 ds &= \int_0^t \mu(\rho_\nu P_s |\psi|^2) \\ &\leq \mu(\rho_\nu^q)^{\frac{1}{q}} \int_0^t \left[\mu(P_s |\psi|^{\frac{2q}{q-1}}) \right]^{1-\frac{1}{q}} ds \\ &= \mu(\rho_\nu^q)^{\frac{1}{q}} \left[\mu(|\psi|^{\frac{2q}{q-1}}) \right]^{1-\frac{1}{q}} < \infty. \end{aligned}$$

Then for $\nu = \delta_x$ or $\nu \in \mathcal{U}$, M_t^ν is a well defined supermartingale. It suffices to prove $\mathbb{E}M_t^\nu = 1$ for any $t \geq 0$. Since $\mathbb{E}M_t^\nu = \int_{\mathbb{R}^d} (\mathbb{E}M_t^x) \nu(dx)$, it remains to show that $(M_t^x)_{t \geq 0}$ is a martingale for any $x \in \mathbb{R}^d$. By the Markov property and the Girsanov theorem, this follows if we can find a constant $t_0 > 0$ such that the Novikov condition

$$(2.10) \quad \mathbb{E} \exp \left(\frac{1}{2} \int_0^{t_0} |\psi(X_s^x)|^2 ds \right) < \infty, \quad \forall x \in \mathbb{R}^d.$$

Indeed, (2.10) implies that $(M_t^x)_{0 \leq t \leq t_0}$ is a martingale for all $x \in \mathbb{R}^d$, so that by the Markov property, for any $s \geq 0$ the process $(M_{s,t}^x)_{t \in [s, s+t_0]} := (\frac{M_t^x}{M_s^x})_{t \in [s, s+t_0]}$ is a martingale under the conditional probability given X_s^x . Thus, by induction and the Markov property we prove that $(M_t^x)_{t \geq 0}$ is a martingale for all $x \in \mathbb{R}^d$ as follows: if $(M_t^x)_{0 \leq t \leq nt_0}$ is a martingale for some $n \geq 1$, then for any $nt_0 \leq s < t \leq (n+1)t_0$ we have

$$\mathbb{E}(M_t^x | \mathcal{F}_s) = M_s^x \mathbb{E}(M_{s,t}^x | \mathcal{F}_s) = M_s^x \mathbb{E}(M_{s,t}^x | X_s^x) = M_s^x.$$

To prove (2.10), we take $t_0 = \frac{\varepsilon}{d}$. By taking $p = 2d$ in Proposition 2.1(4) and using Jensen's inequality we obtain

$$\begin{aligned} \mathbb{E} \exp \left(\frac{1}{2} \int_0^{t_0} |\psi(X_s^x)|^2 ds \right) &\leq \frac{1}{t_0} \int_0^{t_0} P_s e^{\frac{t_0}{2} |\psi|^2}(x) ds \\ &\leq C(t_0) H(x) \left[\mu(e^{dt_0 |\psi|^2}) \right]^{\frac{1}{2d}} < \infty \end{aligned}$$

for some constant $C(t_0) > 0$. □

3 Proofs of Theorems 1.1 and 1.2

We will prove the following more general result Theorem 3.1, which implies Theorem 1.1 for $S_t^\nu := t \{ \mathcal{R}_t(X_{[0,t]}^\nu) - \mathcal{R} \}$ according to Proposition 2.1.

In general, let X_t be a time-homogenous continuous Markov process on \mathbb{R}^d with respect to the filtration \mathcal{F}_t such that the associated Markov semigroup P_t has a unique invariant probability measure μ . Let $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be measurable such that $\mu(|\psi|^p) < \infty$ for any $p > 1$. Since μ is P_t -invariant, for any $\nu \in \mathcal{U}$ and any $q > 1$, the process X_s^ν starting at distribution ν satisfies

$$\begin{aligned} \int_r^t \mathbb{E} |\psi(X_s^\nu)|^q ds &= \int_r^t \nu(P_s |\psi|^q) ds \\ (3.1) \quad &\leq \int_r^t \mu(\rho_\nu^p)^{\frac{1}{p}} \mu(P_s |\psi|^{\frac{2pq}{p-1}})^{1-\frac{1}{p}} ds \\ &= (t-r) \mu(\rho_\nu^p)^{\frac{1}{p}} \mu(|\psi|^{\frac{2pq}{p-1}})^{1-\frac{1}{p}} < \infty, \quad t \geq r \geq 0. \end{aligned}$$

In particular, the additive functional

$$S_t^\nu := \int_0^t \langle \psi(X_s^\nu), dW_s \rangle + \int_0^t \{ |\psi(X_s^\nu)|^2 - \mu(|\psi|^2) \} ds, \quad t \geq 0, \nu \in \mathcal{U}$$

is well defined.

Theorem 3.1. *In the above framework, let $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be measurable such that $\mu(e^{\varepsilon|\psi|^2}) < \infty$ for some constant $\varepsilon > 0$. If P_t is hyperbounded and there exists $t > 0$ such that 1 is a simple eigenvalue of P_t in $L^2(\mu)$, then the following assertions hold:*

- (1) $\delta := \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} |S_t^\mu|^2 < \infty$ exists.
- (2) For any $p > 1$ and $l > 0$, $\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{S_t^\nu}{\sqrt{t}} \in \cdot \right) = N(0, \delta)$ weakly and uniformly in $\nu \in U_\mu^p(l)$.
- (3) For any $\lambda : (0, \infty) \rightarrow (0, \infty)$ with $\lambda(t) \wedge \frac{\sqrt{t}}{\lambda(t)} \rightarrow \infty$ as $t \rightarrow \infty$, any measurable set $A \subset \mathbb{R}$ and constants $p > 1, l > 0$,

$$\begin{aligned} - \inf_{x \in A^c} \frac{x^2}{2\delta} &\leq \liminf_{t \rightarrow \infty} \frac{1}{\lambda(t)^2} \log \mathbb{P} \left(\frac{S_t^\nu}{\lambda(t)\sqrt{t}} \in A \right) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{\lambda(t)^2} \log \mathbb{P} \left(\frac{S_t^\nu}{\lambda(t)\sqrt{t}} \in A \right) \leq - \inf_{x \in A} \frac{x^2}{2\delta} \end{aligned}$$

holds uniformly in $\nu \in U_\mu^p(l)$.

- (4) If $\mu(|\cdot|^p) < \infty$ for any $p > 1$, then for any $\nu \in \mathcal{P}(\mathbb{R}^d)$ with $\rho_\nu := \frac{d\nu}{d\mu} \in L^q(\mu)$ for some $q > 1$, \mathbb{P} -a.s.

$$\limsup_{t \rightarrow \infty} \frac{S_t^\nu}{\sqrt{2t \log \log t}} \leq \sqrt{\delta}, \quad \liminf_{t \rightarrow \infty} \frac{S_t^\nu}{\sqrt{2t \log \log t}} \geq -\sqrt{\delta}.$$

If moreover P_t has density $p_t(x, \cdot)$ with respect to μ such that

$$(3.2) \quad \sup_{l \geq 1} \sup_{|x| \leq l^{r_0}} \mu(p_l(x, \cdot)^q) < \infty$$

holds for some $q > 1$ and $r_0 > 0$, then the equalities hold.

Proof. (a) According to [20, Theorem 2.4'], the assertions (1)-(3) hold provided there exist two constants $t_0, \varepsilon > 0$ such that $\sup_{t \in [0, t_0]} \mathbb{E} e^{\varepsilon |S_t^\mu|} < \infty$. For any $r > 0$ we have

$$(3.3) \quad \mathbb{E} e^{r|S_t^\mu|} \leq \mathbb{E} e^{rS_t^\mu} + \mathbb{E} e^{-rS_t^\mu}.$$

Since $\mu(e^{\varepsilon|\psi|^2}) < \infty$, by Schwartz's and Jensen's inequalities, and noting that μ is the invariant probability measure, for $t_0 := \frac{\varepsilon}{4} > 0$ we have

$$\begin{aligned} \left(\mathbb{E}e^{S_t^\mu}\right)^2 &\leq \mathbb{E} \exp \left(2 \int_0^t \langle \psi(X_s^\mu), dW_s \rangle - 2 \int_0^t |\psi(X_s^\mu)|^2 ds \right) \\ &\quad \times \mathbb{E} \exp \left(4 \int_0^t |\psi(X_s^\mu)|^2 ds - 2t\mu(|\psi|^2) \right) \\ &\leq \frac{1}{t} \int_0^t \mathbb{E} \exp (4s|\psi(X_s^\mu)|^2) ds \\ &\leq \mu(e^{4t_0|\psi|^2}) < \infty, \quad t \in [0, t_0]. \end{aligned}$$

The same estimate holds for $\mathbb{E}e^{-S_t^\mu}$. Then $\sup_{t \in [0, t_0]} \mathbb{E}e^{|S_t^\mu|} < \infty$ for some constant $t_0 > 0$.

(b) To prove (4), we need the following assertion: for any $\nu \in \mathcal{U}$ and $p \geq 2$, there exists a constant $c > 0$ such that

$$(3.4) \quad \mathbb{E}|S_t^\nu - S_s^\nu|^p \leq c(t-s)^{\frac{p}{2}} [\log \log(t-s+e^2)]^{\frac{p}{2}}, \quad t \geq s \geq 0.$$

By (3.1), it suffices to prove the estimate for $t-s \geq e^2$.

We first consider the case that $\nu = \mu$. In this case we only need to consider $s = 0$ due to the stationary property. Since μ is P_t -invariant and $\mu(|\psi|^q) < \infty$ for any $q > 1$, for any $p \geq 2$ there exists a constant $c_1 > 0$ such that

$$(3.5) \quad \mathbb{E}|S_t^\mu|^p \leq c_1 t^p, \quad t \geq e^2.$$

Applying Theorem 1.1(3) to $\lambda(t) := \sqrt{(1+\delta)p \log t}$ for $t \geq e$, we obtain

$$\mathbb{P}(|S_t^\mu| > \lambda(t)\sqrt{2t}) \leq c_2 e^{-p \log t} = c_2 t^{-p}, \quad t \geq e$$

for some constant $c_2 > 0$. Combining this with (3.5) we arrive at

$$\begin{aligned} (3.6) \quad \mathbb{E}|S_t^\mu|^p &\leq \mathbb{E} \left(|S_t^\mu|^p 1_{\{|S_t^\mu| > \lambda(t)\sqrt{2t}\}} \right) + (2t\lambda(t)^2)^{\frac{p}{2}} \\ &\leq \sqrt{\mathbb{P}(|S_t^\mu| > \lambda(t)\sqrt{2t}) \mathbb{E}|S_t^\mu|^{2p}} + c_3 t^{\frac{p}{2}} \lambda(t)^p \leq c_4 t^{\frac{p}{2}} (\log t)^{\frac{p}{2}}, \quad t \geq e \end{aligned}$$

for some constants $c_3, c_4 > 0$. Moreover, applying Theorem 1.1(3) to $\lambda(t) := \sqrt{(1+\delta)p \log \log t}$ for $t \geq e^2$, we obtain

$$\mathbb{P}(|S_t^\mu| > \lambda(t)\sqrt{2t}) \leq c_5 e^{-p \log \log t} = c_5 (\log t)^{-p}, \quad t \geq e^2$$

for some constant $c_5 > 0$. Combining this with (3.6) that $\mathbb{E}|S_t^\mu|^{2p} \leq ct^p (\log t)^p$ for some constant $c > 0$ and $t \geq e$, we arrive at

$$\begin{aligned} \mathbb{E}|S_t^\mu|^p &\leq \mathbb{E} \left(|S_t^\mu|^p 1_{\{|S_t^\mu| > \lambda(t)\sqrt{2t}\}} \right) + (2t\lambda(t)^2)^{\frac{p}{2}} \\ &\leq \sqrt{\mathbb{P}(|S_t^\mu| > \lambda(t)\sqrt{2t}) \mathbb{E}|S_t^\mu|^{2p}} + c_6 t^{\frac{p}{2}} (\log \log t)^{\frac{p}{2}} \leq c_7 t^{\frac{p}{2}} (\log \log t)^{\frac{p}{2}}, \quad t \geq e^2 \end{aligned}$$

for some constants $c_5, c_6, c_7 > 0$. Thus, the assertion holds for $\nu = \mu$.

Next, let $\nu \in \mathcal{U}$ with $\mu(\rho_\nu^q) < \infty$ for some $q > 1$. By the estimate on $\mathbb{E}|S_t^\mu - S_s^\mu|^p$ we have $\mathbb{E}|S_t^x - S_s^x|^p < \infty, \mu$ -a.e. x , where $S_t^\nu := S_t^\mu$ for $\nu = \delta_x$. Moreover,

$$\begin{aligned} \mathbb{E}|S_t^\nu - S_s^\nu|^p &= \int_{\mathbb{R}^d} \rho_\nu(x) \mathbb{E}|S_t^x - S_s^x|^p \mu(dx) \\ &\leq [\mu(\rho_\nu^q)]^{\frac{1}{q}} \left[\mu \left(\mathbb{E}|S_t^x - S_s^x|^p \right)^{\frac{q}{q-1}} \right]^{\frac{q-1}{q}} \\ &\leq [\mu(\rho_\nu^q)]^{\frac{1}{q}} \left[\mu \left(\mathbb{E}|S_t^x - S_s^x|^{\frac{pq}{q-1}} \right) \right]^{\frac{q-1}{q}} \\ &= [\mu(\rho_\nu^q)]^{\frac{1}{q}} \left(\mathbb{E}|S_t^\mu - S_s^\mu|^{\frac{pq}{q-1}} \right)^{\frac{q-1}{q}} \\ &\leq c(t-s)^{\frac{p}{2}} \{\log \log(t-s+e^2)\}^{\frac{p}{2}} \end{aligned}$$

holds for some constant $c > 0$.

(c) To prove (4), we will take a sequence $t_n \uparrow \infty$ to replace the continuous limit for $t \uparrow \infty$. Unlike the standard choice $t_n = p^n$ for $p > 1$ in the literature (see [7]), we take $t_n = e^{n^\theta}$ for some $\theta \in (0, 1)$ where $\theta < 1$ is crucial in the argument.

Since $\theta < 1$, we may take $p > 1$ such that $p(1-\theta) > 1$. For any $\varepsilon > 0$, by the stationary property of the process, the Burkhold inequality and (3.4), we obtain

$$\begin{aligned} \mathbb{P} \left(\max_{t \in [t_n, t_{n+1}]} |S_t^\nu - S_{t_n}^\nu| > \varepsilon \sqrt{2t_n \log \log t_n} \right) &\leq \frac{c_1 \mathbb{E}|S_{t_{n+1}}^\nu - S_{t_n}^\nu|^{2p}}{(2\varepsilon^2 t_n \log \log t_n)^p} \\ &\leq \frac{c_2 (t_{n+1} - t_n)^p [\log \log(t_{n+1} + e^2)]^p}{t_n^p (\log \log t_n)^p} \leq c_3 n^{(\theta-1)p}, \quad n \geq 2 \end{aligned}$$

for some constants $c_1, c_2, c_3 > 0$. Hence,

$$\mathbb{P} \left(\max_{t \in [t_n, t_{n+1}]} \frac{|S_t^\nu - S_{t_n}^\nu|}{\sqrt{2t_n \log \log t_n}} > \varepsilon \right) \leq c_3 n^{(\theta-1)p}, \quad n \geq 2.$$

Since $p(1-\theta) > 1$, this implies

$$\sum_{n=2}^{\infty} \mathbb{P} \left(\max_{t \in [t_n, t_{n+1}]} \frac{|S_t^\nu - S_{t_n}^\nu|}{\sqrt{2t_n \log \log t_n}} > \varepsilon \right) < \infty,$$

so that by the Borel-Cantelli lemma,

$$\limsup_{n \rightarrow \infty} \max_{t \in [t_n, t_{n+1}]} \frac{|S_t^\nu - S_{t_n}^\nu|}{\sqrt{2t_n \log \log t_n}} \leq \varepsilon, \quad \text{a.s.}$$

By the arbitrariness of $\varepsilon > 0$, we arrive at

$$(3.7) \quad \limsup_{n \rightarrow \infty} \max_{t \in [t_n, t_{n+1}]} \frac{|S_t^\nu - S_{t_n}^\nu|}{\sqrt{2t_n \log \log t_n}} = 0, \quad \text{a.s.}$$

(d) We now prove assertion (4) for $\delta = 0$. In this case, for any $\varepsilon > 0$, Theorem 3.1(3) implies

$$\lim_{n \rightarrow \infty} \frac{1}{\log \log t_n} \log \mathbb{P} \left(\frac{|S_{t_n}^\nu|}{\sqrt{2t_n \log \log t_n}} > \varepsilon \right) = -\infty,$$

so that we may find a constant $c > 0$ such that

$$\mathbb{P} \left(\frac{|S_{t_n}^\nu|}{\sqrt{2t_n \log \log t_n}} > \varepsilon \right) \leq c \exp \left[-\frac{2}{\theta} \log \log t_n \right] = \frac{c}{n^2}, \quad n \geq 1.$$

Then $\sum_{n=1}^{\infty} \mathbb{P} \left(\frac{|S_{t_n}^\nu|}{\sqrt{2t_n \log \log t_n}} > \varepsilon \right) < \infty$. By the Borel-Cantelli lemma, this implies

$$\limsup_{n \rightarrow \infty} \frac{|S_{t_n}^\nu|}{\sqrt{2t_n \log \log t_n}} \leq \varepsilon, \quad \text{a.s.}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{n \rightarrow \infty} \frac{|S_{t_n}^\nu|}{\sqrt{2t_n \log \log t_n}} = 0, \quad \text{a.s.}$$

Combining this with (3.7) we prove (4) for $\delta = 0$.

(e) Let $\delta \in (0, \infty)$. In this case by using $\delta^{-1/2}\psi$ to replace ψ , we may and do assume that $\delta = 1$. We will only prove the first limit as that of the second is completely similar. We first prove the upper bound estimate

$$(3.8) \quad \limsup_{t \rightarrow \infty} \frac{S_t^\nu}{\sqrt{2t \log \log t}} \leq 1.$$

For any $r \in (0, 1)$, take $\theta \in ((1+r)^{-2}, 1)$ and $t_n = e^{n^\theta}$ for $n \geq 1$. We have $t_{n+1} - t_n \leq c_1 t_n n^{\theta-1}$ for some constant $c_1 > 0$. By Theorem 3.1(3) with $\delta = 1$ and $\lambda(t) = \sqrt{\log \log t}$ for large $t > 0$, we obtain

$$\mathbb{P} \left(S_{t_n}^\nu \geq (1+2r)\sqrt{2t_n \log \log t_n} \right) \leq c_2 \exp \left[-(1+r)^2 \log \log t_n \right] = c_2 n^{-\theta(1+r)^2}, \quad n \geq 2$$

for some constant $c_2 > 0$. Since $\theta(1+r)^2 > 1$, this implies

$$\sum_{n=2}^{\infty} \mathbb{P} \left(S_{t_n}^\nu \geq (1+2r)\sqrt{2t_n \log \log t_n} \right) < \infty,$$

so that by the Borel-Cantelli lemma,

$$\limsup_{n \rightarrow \infty} \frac{S_{t_n}^\nu}{\sqrt{2t_n \log \log t_n}} \leq 1+2r, \quad \text{a.s.}$$

Combining this with (3.7) we obtain

$$\limsup_{t \rightarrow \infty} \frac{S_t^\nu}{\sqrt{2t \log \log t}} \leq 1+2r, \quad \text{a.s.}$$

Since $r > 0$ is arbitrary, we prove (3.8).

It remains to prove the following lower bound estimate for $\delta = 1$ under condition (3.2):

$$(3.9) \quad \limsup_{t \rightarrow \infty} \frac{S_t^\nu}{\sqrt{2t \log \log t}} \geq 1, \quad \text{a.s.}$$

For any $\varepsilon \in (0, \frac{1}{2})$ and $p \geq 2$, we have

$$\begin{aligned} & \lim_{l \rightarrow \infty} \frac{(1 - 2\varepsilon)\sqrt{2p^l \log \log p^l} - (1 - \varepsilon)\sqrt{2(p^l - p^{l-1} - l) \log \log(p^l - p^{l-1} - l)}}{\sqrt{2(p^{l-1} + l) \log \log(p^{l-1} + l)}} \\ &= (1 - 2\varepsilon)\sqrt{p} - (1 - \varepsilon)\sqrt{p - 1}, \end{aligned}$$

which goes to $-\infty$ as $p \rightarrow \infty$. Then we may find constants $p, l_0 \geq 2$ such that

$$(3.10) \quad \begin{aligned} & (1 - 2\varepsilon)\sqrt{2p^l \log \log p^l} - (1 - \varepsilon)\sqrt{2(p^l - p^{l-1} - l) \log \log(p^l - p^{l-1} - l)} \\ & \leq -2\sqrt{2(p^{l-1} + l) \log \log(p^{l-1} + l)}, \quad l \geq l_0. \end{aligned}$$

Let

$$G_l = \{S_{p^l}^\nu \geq (1 - 2\varepsilon)\sqrt{2p^l \log \log p^l}\}, \quad l \geq l_0.$$

We aim to prove

$$(3.11) \quad \mathbb{P}\left(\bigcap_{n=l_0}^{\infty} \bigcup_{m=n}^{\infty} G_m\right) = 1,$$

which implies

$$\limsup_{n \rightarrow \infty} \frac{S_{p^n}^\nu}{\sqrt{2p^n \log \log p^n}} \geq 1 - 2\varepsilon,$$

so that

$$\limsup_{t \rightarrow \infty} \frac{S_t^\nu}{\sqrt{2t \log \log t}} \geq 1 - 2\varepsilon.$$

By the arbitrariness of $\varepsilon \in (0, \frac{1}{2})$ this implies the desired estimate (3.9).

We now prove (3.11). For any $l \geq l_0 + 1$, by (3.10) we have

$$G_l^c \subset H_{l,1} \cup H_{l,2}$$

for

$$(3.12) \quad \begin{aligned} H_{l,1} &:= \left\{S_{p^{l-1}+l}^\nu \leq -2\sqrt{2(p^{l-1} + l) \log \log(p^{l-1} + l)}\right\}, \\ H_{l,2} &:= \left\{S_{p^l}^\nu - S_{p^{l-1}+l}^\nu \leq (1 - \varepsilon)\sqrt{2(p^l - p^{l-1} - l) \log \log(p^l - p^{l-1} - l)}\right\}. \end{aligned}$$

Hence, for any integer numbers $l > n \geq l_0$, by Markov property we have

$$\begin{aligned}
(3.13) \quad \mathbb{P}\left(\bigcap_{m=n}^l G_m^c\right) &\leq \mathbb{P}\left(\{H_{l,1} \cup H_{l,2}\} \cap \left\{\bigcap_{m=n}^{l-1} G_m^c\right\}\right) \\
&\leq \mathbb{P}(H_{l,1}) + \mathbb{E}\left[\mathbb{P}(H_{l,2} | \mathcal{F}_{p^{l-1}}) \prod_{m=n}^{l-1} 1_{G_m^c}\right] \\
&= \mathbb{P}(H_{l,1}) + \mathbb{E}\left[\mathbb{P}(H_{l,2} | X_{p^{l-1}}^\nu) \prod_{m=n}^{l-1} 1_{G_m^c}\right].
\end{aligned}$$

By Theorem 3.1(3) with $\lambda(t) = \sqrt{\log \log t}$ for large $t > 0$, there exist constants $c_1, c_2 > 0$ such that for $l \geq l_0$,

$$(3.14) \quad \mathbb{P}(H_{l,1}) \leq c_1 \exp[-3 \log \log(p^{l-1} + l)] \leq c_2 l^{-3}.$$

Let $P_t(x, \cdot)$ be the distribution of X_t^x .

$$\begin{aligned}
(3.15) \quad &\mathbb{E}\left[\mathbb{P}(H_{l,2} | X_{p^{l-1}}^\nu) \prod_{m=n}^{l-1} 1_{G_m^c}\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(\frac{S_{p^l}^\nu - S_{p^{l-1}+l}^\nu}{\sqrt{2(p^l - p^{l-1} - l) \log \log(p^l - p^{l-1} - l)}} \leq 1 - \varepsilon \mid X_{p^{l-1}}^\nu\right) \prod_{m=n}^{l-1} 1_{G_m^c}\right] \\
&\leq \mathbb{P}(|X_{p^{l-1}}^\nu| \geq l^{r_0}) \\
&\quad + \mathbb{E}\left[\prod_{m=n}^{l-1} 1_{G_m^c}\right] \sup_{|x| \leq l^{r_0}} \mathbb{P}\left(\frac{S_{p^l - p^{l-1} - l}^{P_l(x, \cdot)}}{\sqrt{2(p^l - p^{l-1} - l) \log \log(p^l - p^{l-1} - l)}} \leq 1 - \varepsilon\right),
\end{aligned}$$

where the last step follows from the time-homogenous Markov property that given $X_{p^{l-1}}^\nu = x$, the conditional distribution of $X_{p^{l-1}+l}^\nu$ is $P_l(x, \cdot)$, and the conditional distribution of $S_{p^l - p^{l-1} - l}^\nu$ coincides with the distribution of $S_{p^l - p^{l-1} - l}^{P_l(x, \cdot)}$. By the condition in (4) we have

$$\begin{aligned}
(3.16) \quad \mathbb{P}(|X_{p^{l-1}}^\nu| \geq l^{r_0}) &\leq l^{-2} \mathbb{E}|X_{p^{l-1}}^\nu|^{2/r_0} \\
&= \frac{1}{l^2} \int_{\mathbb{R}^d} \rho_\nu(x) P_{p^{l-1}}|\cdot|^{2/r_0}(x) \mu(dx) \\
&\leq \frac{1}{l^2} [\mu(\rho_\nu^q)]^{\frac{1}{q}} \left[\mu(|\cdot|^{2q/(q-1)})\right]^{\frac{q-1}{q}} \leq \frac{c_3}{l^2}
\end{aligned}$$

for some constant $c_3 > 0$. Moreover, by (3.2) and Theorem 3.1(3) with $\lambda(t) = \log \log t$ for large t , we have

$$\begin{aligned}
&\sup_{|x| \leq l^{r_0}} \mathbb{P}\left(S_{p^l - p^{l-1} - l}^{P_l(x, \cdot)} \leq (1 - \varepsilon) \sqrt{2(p^l - p^{l-1} - l) \log \log(p^l - p^{l-1} - l)}\right) \\
&= 1 - \inf_{|x| \leq l^{r_0}} \mathbb{P}\left(S_{p^l - p^{l-1} - l}^{P_l(x, \cdot)} > (1 - \varepsilon) \sqrt{2(p^l - p^{l-1} - l) \log \log(p^l - p^{l-1} - l)}\right) \\
&\leq 1 - \exp\left[-\left(1 - \frac{\varepsilon}{2}\right)^2 \log \log(p^l - p^{l-1} - l)\right] \leq 1 - (l \log p)^{-(1 - \frac{\varepsilon}{2})^2}, \quad l \geq l_0 + 1.
\end{aligned}$$

Combining this with (3.15) and (3.16), we get

$$(3.17) \quad \mathbb{E} \left[\mathbb{P} \left(H_{l,2} \middle| X_{p^{l-1}}^\nu \right) \prod_{m=n}^{l-1} 1_{G_m^c} \right] \leq \frac{c_3}{l^2} + \left(1 - (l \log p)^{-(1-\frac{\varepsilon}{2})^2} \right) \mathbb{E} \left(\prod_{m=n}^{l-1} 1_{G_m^c} \right),$$

which, together with (3.13) and (3.14), yields

$$\mathbb{P} \left(\bigcap_{m=n}^l G_m^c \right) \leq \frac{c}{l^2} + (1 - (l \log p)^{-(1-\frac{\varepsilon}{2})^2}) \mathbb{P} \left(\bigcap_{m=n}^{l-1} G_m^c \right), \quad l \geq n+1 \geq l_0+1$$

for some constants $c > 0$. Therefore, by induction we obtain

$$\mathbb{P} \left(\bigcap_{m=n}^l G_m^c \right) \leq c \sum_{m=n}^l m^{-2} + \prod_{m=n+1}^l (1 - (m \log p)^{-(1-\frac{\varepsilon}{2})^2}), \quad l \geq n+1 \geq l_0+1.$$

This implies

$$\mathbb{P} \left(\bigcap_{m=n}^{\infty} G_m^c \right) = \lim_{l \rightarrow \infty} \mathbb{P} \left(\bigcap_{m=n}^l G_m^c \right) \leq c \sum_{m=n}^{\infty} m^{-2},$$

so that

$$\mathbb{P} \left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} G_m \right) = \lim_{n \rightarrow \infty} \mathbb{P} \left(\bigcup_{m=n}^{\infty} G_m \right) = 1 - \lim_{n \rightarrow \infty} \mathbb{P} \left(\bigcap_{m=n}^{\infty} G_m^c \right) = 1.$$

Therefore, (3.11) holds and the proof is finished. \square

Proofs of Theorem 1.1 and Theorem 1.2. Let $\psi = 2\sigma^{-1}B - \sigma^* \nabla \log \rho$. By Proposition 2.1, all conditions in Theorem 3.1 hold, so that Theorem 1.1 follows immediately. To prove Theorem 1.2, we first show that

$$(3.18) \quad \mathbb{E} e^{r|S_{t_0}^x|} \leq H(x), \quad x \in \mathbb{R}^d$$

holds for some constants $r, t_0 > 0$ and locally bounded function H on \mathbb{R}^d . Indeed, writing $e^{r|S_{t_0}^x|} \leq e^{rS_{t_0}^x} + e^{-rS_{t_0}^x}$, (3.18) with $r = 1$ and small $t_0 > 0$ follows from Proposition 2.1(1) and (4) since

$$\mathbb{E} e^{\int_0^{t_0} |\psi(X_s^x)|^2 ds} \leq \frac{1}{t_0} \int_0^{t_0} \mathbb{E} e^{t_0 |\psi(X_s^x)|^2} ds = \frac{1}{t_0} \int_0^{t_0} P_s e^{t_0 |\psi|^2}(x) ds.$$

Now, by the Markov property, $(S_t^x - S_{t_0}^x)_{t \geq t_0}$ identifies with $(S_{t-t_0}^{P_{t_0}(x, \cdot)})_{t \geq t_0}$ in distribution. Moreover, by Proposition 2.1, $\mu(p_{t_0}(x, \cdot)^2)$ is locally bounded in x . So, (1) and (3) in Theorem 1.2 follow from (2) and (4) in Theorem 3.1 respectively. To prove Theorem 1.2(2), we observe that for λ therein it follows from (3.18) that

$$\limsup_{t \rightarrow \infty} \frac{1}{\lambda(t)^2} \log \mathbb{P} \left(\frac{|S_{t_0}^x|}{\lambda(t) \sqrt{2t}} > \varepsilon \right) \leq \limsup_{t \rightarrow \infty} \frac{\log \exp[-r\lambda(t)\sqrt{2t}]}{\lambda(t)^2} = -\infty$$

locally uniformly in x . Then Theorem 1.2(2) follows from Theorem 3.1(3) since $\mu(p_{t_0}(x, \cdot)^2)$ is locally bounded in x . \square

4 SDEs with multiplicative noise

Consider the SDE

$$(4.1) \quad dX_t = B(X_t)dt + \sigma(X_t)dW_t$$

as (1.2), but σ now depends on the space variable x . When B is Lipschitz continuous and $\sigma \in C_b^1(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$ such that $c_1 I \leq \sigma \sigma^* \leq c_2 I$ for some constants $c_2 \geq c_1 > 0$, and the dissipativity condition

$$(4.2) \quad \|\sigma(x) - \sigma(y)\|_{HS}^2 + 2\langle B(x) - B(y), x - y \rangle \leq -K|x - y|^2, \quad x, y \in \mathbb{R}^d$$

holds for some constant $K > 0$, we can prove the existence and uniqueness of invariant probability measure μ with $\mu(e^{|\cdot|^2}) < \infty$ for some $\varepsilon > 0$. Moreover, by [17, Theorem 1.1] the associated Markov semigroup P_t satisfies

$$(4.3) \quad |P_t f(x)|^p \leq (P_t |f|^p(y)) \exp \left[\frac{c(p)|x - y|^2}{e^{Kt} - 1} \right], \quad p > p_0, t > 0, x, y \in \mathbb{R}^d$$

for all $f \in \mathcal{B}_b(\mathbb{R}^d)$, where $p_0 \geq 1$ is a constant depending on the smallest and largest eigenvalues of $\sigma \sigma^*$ and $c(p)$ is a constant depending on p . As in the proof of Proposition 2.1, this together with $\mu(e^{|\cdot|^2}) < \infty$ implies the hyperboundedness of P_t as well as the local boundedness of $\mu(p_t(x, \cdot)^p)$ for some $p > 1$. The only problem for us to extend Theorem 1.1 and Theorem 1.2 to the present setting is that we do not have good enough integration by parts formula to imply $\mu(e^{\varepsilon|\nabla \log \rho|^2}) < \infty$ for some constant $\varepsilon > 0$, where ρ is the density of μ which is again strictly positive and belongs to $\cap_{p>1} W_{loc}^{p,1}(dx)$ according to [3]. Due to this problem, in the moment we are not able to start from a given drift B , but start from a given invariant probability measure μ with the required property $\mu(e^{\varepsilon|\nabla \log \rho|^2}) < \infty$. This can be done by perturbations to symmetric diffusion process.

Now, let $V \in C^2(\mathbb{R}^d)$ such that

$$(4.4) \quad \int_{\mathbb{R}^d} e^{\varepsilon|\nabla V(x)|^2 + V(x)} dx < \infty \text{ for some constant } \varepsilon > 0.$$

Without loss of generality, we may and do assume that $\mu(dx) := e^{V(x)} dx$ is a probability measure. Let $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be locally integrable such that

$$(4.5) \quad \int_{\mathbb{R}^d} \langle b, \nabla f \rangle d\mu = 0, \quad f \in C_0^\infty(\mathbb{R}^d).$$

Let $\{e_i\}_{i=1}^d$ be the standard ONB of \mathbb{R}^d . We assume that

$$(4.6) \quad B := b + \frac{1}{2} \sum_{ij=1}^d \{\partial_j(\sigma \sigma^*)_{ij}\} e_i + \frac{1}{2}(\sigma \sigma^*) \nabla V$$

has linear growth. Then μ is the unique invariant probability measure of P_t and, as in the additive noise case, the sample EPR of the solution to (4.1) can be formulated as

$$\mathcal{R}_t(X_{[0,t]}) = \frac{1}{t} \int_0^t \langle \psi(X_s), dW_s \rangle + \frac{1}{2t} \int_0^t |\psi(X_s)|^2 ds,$$

where, noting that $\nabla \rho = V$ in the present setting,

$$\psi := 2\sigma^{-1}B - \sigma\sigma^*\nabla V = 2\sigma^{-1}b + \sum_{i,j=1}^d \{\partial_j(\sigma\sigma^*)_{ij}\}\sigma^{-1}e_i.$$

Since B has linear growth, $c_1I \leq \sigma\sigma^* \leq c_2I$ and $\mu(e^{\varepsilon(|\cdot|^2 + |\nabla V|^2)}) < \infty$ for some constant $\varepsilon > 0$, we have $\mu(e^{\varepsilon|\psi|^2}) < \infty$ for some $\varepsilon > 0$ as required in Theorem 3.1. Therefore, the following result follows from Theorem 3.1.

Theorem 4.1. *Let $\sigma \in C_b^1(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$ be such that $c_1I \leq \sigma\sigma^* \leq c_2I$ for some constants $c_2 \geq c_1 > 0$, let $V \in C^2(\mathbb{R}^d)$ satisfy (4.4), and let b satisfy (4.5) such that B defined in (4.6) has linear growth. If (4.2) holds for some constant $K > 0$, then all assertions in Theorems 1.1 and 1.2 hold for $\log \rho = V$.*

Example 4.1. A simple example satisfying all conditions in Theorem 4.1 is that $V(x) = \alpha - \beta|x|^2$ for constants $\alpha \in \mathbb{R}$ and $\beta > 0$ such that $e^{V(x)}dx$ is a probability measure, $b(x) = Ax$ for some antisymmetric $d \times d$ -matrix A , $\sigma \in C_b^1(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$ such that $c_1I \leq \sigma\sigma^* \leq c_2I$ for some constants $c_2 \geq c_1 > 0$, and

$$(4.7) \quad \lambda < \frac{1}{2c_2} \left(d\|\nabla \sigma\|_\infty^2 + 2\|A\| + \left\{ \sum_{i=1}^d \left(\sum_{j=1}^d \|\partial_j(\sigma\sigma^*)_{ij}\|_\infty \right)^2 \right\}^{\frac{1}{2}} \right),$$

which implies (4.2) for some constant $K > 0$.

Below we use the log-Sobolev inequality to replace the condition (4.2). As observed in the proof of Proposition 2.1(3) that if P_t has an invariant probability measure μ then 1 is a simple eigenvalue of P_t for $t > 0$. So, the key condition in Theorem 3.1 is the hyperboundedness of P_t . We will see that this follows from the following log-Sobolev inequality in the uniformly elliptic case:

$$(4.8) \quad \mu(f^2 \log f^2) \leq C\mu(|\nabla f|^2), \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f^2) = 1$$

for a constant $C > 0$. Due to the Bakry-Emery criterion [2], this inequality holds with $C = \frac{2}{K}$ if $\text{Hess}_V \leq -K$ for some constant $K > 0$. By [5] the log-Sobolev inequality (4.8) holds if $\text{Hess}_V(x) \leq -K$ for some constant $K > 0$ and large enough $|x| > 0$. In case that $\text{Hess}_V(x) \leq K$ for some positive constant $K > 0$ and large enough $|x|$, according to [15] the log-Sobolev inequality holds provided $\mu(e^{\varepsilon|\cdot|^2}) < \infty$ holds for some $\lambda > \frac{K}{2}$. See also [4] for Lyapunov type sufficient conditions of the log-Sobolev inequality.

Now, we state the following alternative version of Theorem 4.1 with condition (4.2) replaced by (4.8). This result applies to Example 4.1 without assuming (4.7). The price we have to pay is that we can not prove the exact LIL due to the lack of the moment estimate (2.1) on the heat kernel.

Theorem 4.2. *Let $\sigma \in C_b^1(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$ such that $c_1 I \leq \sigma \sigma^* \leq c_2 I$ for some constants $c_2 \geq c_1 > 0$, let $V \in C^1(\mathbb{R}^d)$ such that $\mu(dx) := e^{V(x)} dx$ is a probability measure satisfying (4.4) and (4.8), and let b satisfy (4.5) and has linear growth. Then for the SDE with B given by (4.6), all assertions in Theorems 1.1 (1)-(3) hold, and for any $\nu \in \mathcal{P}(\mathbb{R}^d)$ with $\rho_\nu := \frac{d\nu}{d\mu} \in L^q(\mu)$ for some $q > 1$, \mathbb{P} -a.s.*

$$\limsup_{t \rightarrow \infty} \frac{S_t^\nu}{\sqrt{2t \log \log t}} \leq \sqrt{\delta}, \quad \liminf_{t \rightarrow \infty} \frac{S_t^\nu}{\sqrt{2t \log \log t}} \geq -\sqrt{\delta}.$$

Proof. Since σ is C^1 -smooth and B has linear growth, the SDE (4.1) has a unique non-explosive solution. Let P_t be the associated Markov semigroup. By (4.8), $\mu(e^{\varepsilon|\cdot|^2}) < \infty$ holds for some constant $\varepsilon > 0$ (see [1]). Since $\psi := 2\sigma^{-1}B - \sigma^* \nabla V$, B has linear growth and $c_1 I \leq \sigma \sigma^* \leq c_2 I$, this and (4.4) imply $\mu(e^{\varepsilon|\psi|^2}) < \infty$ for some $\varepsilon > 0$.

Moreover, as explained in the proof of Proposition 2.1(3) using [12, Proposition 2.11], (4.5), (4.6) and $b \in L^p(\mu)$ for all $p > 1$ implies that P_t is associated to a Dirichlet form with symmetric part

$$\mathcal{E}(f, g) := \mu(\langle \sigma \sigma^* \nabla f, \nabla g \rangle), \quad f, g \in H_\sigma^{2,1}(\mu),$$

and 1 is a simple eigenvalue of P_t for $t > 0$. By (4.8) we have

$$(4.9) \quad \mu(f^2 \log f^2) \leq \frac{C}{c_1} \mathcal{E}(f, f), \quad f \in H_\sigma^{2,1}(\mu), \mu(f^2) = 1.$$

So, according to [9], the semigroup P_t is hypercontractive. Then the proof is finished by Theorem 3.1. \square

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